Two-scale extensions for non-periodic coefficients *

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Abstract

We consider non-homogeneous media with properties which can be characterized by rapidly oscillated coefficients. For such coefficients we define a notion of two-scale extension, present several ways to construct two-scale extensions, discuss their properties and relation to homogenization

Key words. homogenization, non-periodic coefficients, two-scale convergence, admissible test functions, elliptic equation.

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1 Introduction

It is usually difficult to predict a global behaviour of some process in heterogeneous media (for example composite/porous materials) although the physics of the process might be well understood locally. The reason lying in the complexity of the microstructure gives rise to different upscaling methods.

Heterogeneities having periodic microstructure play a central role in the development of upscaled models. From one side they represent an important particular case of general heterogeneous media and on the other there are well developed mathematical techniques (e.g. the two-scale asymptotic expansion method), which help to derive formally and often rigorously the upscaled model. As a result many physical processes in heterogeneous media having periodic microstructures are well investigated both from theoretical and from practical points of view and the periodicity assumption is usually a starting point for the upscaling procedures [2],[13],[15]. Although this assumption is valid in only limited number of cases, mostly in artificially created materials. Therefore for practical purposes one should be able to deal with non- periodic structures.

The deterministic homogenization procedure starts from a sequence of problems $\{\mathcal{P}^\varepsilon\}$. In the periodic case the heterogeneity in $\mathcal{P}^\varepsilon$ is usually described by an ε-periodic function $a^\varepsilon(x) = a(x/\varepsilon)$, where $a(y)$ is a given Y-periodic function in $\mathbb{R}^d$ ($Y = (0,1)^d$ is a period: $a(y + e_i) = a(y)$, e_i is a unit vector, $i = 1, \dots, d$). Quite often the purely periodic coefficient can be generalized without difficulties to the locally periodic coefficient $a^\varepsilon(x) = a(x, x/\varepsilon)$ (where $a(x, y)$ is a given Y-periodic function in y). In the following steps one has to investigate the convergence of the sequence (in a wide sense) and to find a limit problem $\mathcal{P}^0$. The solution of the limit problem can be used in order to approximate the solutions of the problems $\mathcal{P}^\varepsilon$ for small enough ε.

The coefficients $a(y)$ or $a(x, y)$ are considered in mathematical literature as given functions belonging to some functional spaces, without paying much attention where they come from. The

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construction of these coefficients which is important for usage of homogenization results will be discussed in this article.

Let us assume that some process in a heterogeneous medium occupying a bounded domain $\Omega \subset \mathbb{R}^d$ can be described by some PDE(s) with (at least one) rapidly oscillated coefficient $a_M(x)$, which is non necessarily periodic. This is our initial problem $\mathcal{P}$.

Asymptotical approach applied to $\mathcal{P}$ means that we are not going to solve it directly, but to construct a sequence of imaginary problems $\{\mathcal{P}^{\varepsilon}\}$ passing through $\mathcal{P}$ at some $\bar{\varepsilon}$:

$$\begin{array}{ccccccccc} \mathcal{P}^{\varepsilon_0}, & \dots & ,\mathcal{P}^{\varepsilon_{n-1}}, & \mathcal{P}^{\bar{\varepsilon}}, & \mathcal{P}^{\varepsilon_{n+1}}, & \dots & ,\mathcal{P}^{\varepsilon}, & \dots & \dashrightarrow \ \mathcal{P}^0. \\ & & & \| & & & & & \\ & & & \mathcal{P} & & & & & \end{array} \tag{1}$$

If the sequence $\{\mathcal{P}^{\varepsilon}\}$ is convergent in some sense to a limit problem $\mathcal{P}^0$ which is easier than $\mathcal{P}$ then the solution of $\mathcal{P}^0$ can be used to approximate (in some sense) the solutions of $\mathcal{P}^{\varepsilon}$, and in particular, of $\mathcal{P}^{\bar{\varepsilon}}$ (it is our main goal). The "convergence of problems" is related to convergence of their solutions, but it might be restrictive to say something more precise.

In the periodic case, namely when $a_M(x)$ is $\bar{\varepsilon}$-periodic in Ω there is a Y-periodic function $a(y)$ defined in $\mathbb{R}^d$ such that $a_M(x) = a(x/\bar{\varepsilon})$. The standard sequence $\{\mathcal{P}^{\varepsilon}\}$ is based on the ε-periodic coefficient $a^{\varepsilon}(x) = a(x/\varepsilon)$. $a^{\bar{\varepsilon}}(x) = a_M(x)$ and consequently the condition $\mathcal{P}^{\bar{\varepsilon}} = \mathcal{P}$ is not difficult to satisfy. This approach cannot be used for non-periodic $a_M(x)$ since there is no such periodic $a(y)$ exists (except the case when the period contains the whole Ω). But using the sequence $\{\mathcal{P}^{\varepsilon}\}$ based on locally periodic function $a(x,y)$, where the coefficients have the form $a^{\varepsilon}(x) = a(x, x/\varepsilon)$, the requirement $\mathcal{P}^{\bar{\varepsilon}} = \mathcal{P}$ becomes much more realizable. We only need to find such function $a(x,y)$ and $\bar{\varepsilon}$ that $a(x, x/\bar{\varepsilon}) = a_M(x)$. Therefore it is reasonable to make the following definition.

Definition 1.1. *Let us say that a function $a(x,y)$, $(x,y) \in \Omega \times \mathbb{R}^d$, Y-periodic in the variable y is a two–scale extension for $a_M(x)$ if there exists a positive number $\bar{\varepsilon}$ such that*

$$a\left(x, \frac{x}{\bar{\varepsilon}}\right) = a_M(x), \qquad \forall x \in \Omega. \tag{2}$$

The article is organized as follows. In the next section several ways to construct two–scale extension for arbitrary initial coefficients $a_M(x)$ are presented. The Section 3 contains a short introduction to the two-scale convergence method together with a definition and a criterion for the concept of admissible test function. The criterion is needed to show that the proposed in Section 2 two scale extensions are admissible test functions in the sense of the two-scale convergence. This is the main purpose of Sections 4, 5, 6 (its justification consists of several results which may also be useful of their own). The application to the second order elliptic equation is discussed in Section 7.

Why do we need this? There are both theoretical and practical reasons to consider two-scale extensions. First of all, they seem to be naturally related to the formal method of two-scale asymptotic expansions and to its rigorous version – the two-scale convergence method. If some mathematical model of a physical process allows the formal homogenization procedure via two-scale asymptotic expansions in the case of smooth locally periodic coefficients then as the next step one can substitute two-scale extensions for these coefficients and check whether the homogenization procedure remains working for non-periodic coefficients.

Let us now assume that our mathematical model is based on the second order elliptic equation. The two-scale extensions might be useful for better understanding of the following important questions related to the concept of the averaged coefficient:

– its definition, existence, properties, limits of applicability, averaging size;

– connection between deterministic and stochastic approaches;
– reiterative averaging (averaging of the averaged coefficient).

There are many algorithms currently known for practical calculation of the averaged coefficient (see e.g. [3],[6],[14]). Some of them (having the same local problem with periodic boundary conditions) can be recovered by a special choice of the two-scale extension. This gives them a justification by an asymptotical argument as well as some freedom for improvement and generalization. For example it is possible to correct the averaged solution in a postprocessing step using a standard technique from homogenization theory [2, p.76]. Therefore for the practical problems like heat transfer in composite materials and unsaturated flow in heterogeneous porous media the choice of the two-scale extension defines a numerical method which can be used as a possible alternative to such methods as multiscale finite element method [7],[8] or heterogeneous multiscale method [5].

2 Three approaches to construct a two–scale extension

First of all we have the $\mathcal{T}rivial$ Extension:

$$a(x,y) := a_M(x), \qquad x \in \Omega, y \in \mathbb{R}^d.$$

But we cannot expect something better than the constant sequence $\{\mathcal{P}^\varepsilon\} = \{\mathcal{P}\}$ with the limit problem $\mathcal{P}^0 = \mathcal{P}$ which is just as difficult to solve. This practically useless extension gives although an approximation to $\mathcal{P}$ with a perfect quality. Different two-scale extensions lead to upscaled problems with different quality. At least we know that not all are bad.

For the other two approaches we need to know $a_M(x)$ in a neighbourhood of a point in Ω. Since this can create some problems close to the boundary, let us assume that $a_M(x)$ can be somehow extended to a larger domain $\tilde{\Omega}$ which is also bounded (if we find nothing better, we can choose some value of $a_M(\cdot)$ in Ω as a constant value in $\tilde{\Omega} \setminus \Omega$).

Next we need to choose $\bar{\varepsilon}$. For periodic $a_M(x)$ it is reasonable to choose $\bar{\varepsilon}$ equal to the period, but in general we are free in choosing it. Let $W(x)$ be an $\bar{\varepsilon}$–cube with the center x and sides aligned with the coordinate axes. Up to now the only restrictions on $\bar{\varepsilon}$ are: we consider $\bar{\varepsilon}$ to be small comparing to the typical size of Ω and all cubes $W(x)$, $x \in \Omega$ should be completely inside $\tilde{\Omega}$.

Having in mind the volume averaging method it might be reasonable to call $W(x)$ as a (cubic) representative elementary volume (REV) around the point x.

Two approaches to construct the two–scale extension $a(x,y)$ for $a_M(x)$ are different in the sense that the first is created via continuous ($\mathcal{C}ontinuous$ Extension) and the second via discrete ($\mathcal{D}iscrete$ Extension) 'motion' of $W(x)$ in Ω.

2.1 $\mathcal{C}ontinuous$ Extension

Let x be some fixed point in Ω.
• First we define an auxiliary function $\tilde{a}(x,\cdot)$ at $y \in W(x)$:

$$\tilde{a}(x,y) = a_M(y), \qquad y \in W(x).$$

• Secondly we extend it to the whole $\mathbb{R}^d$ periodically – $\tilde{a}(x,y)$ is $\bar{\varepsilon}$-periodic in y.
• Thirdly

$$a(x,y) := \tilde{a}(x, \bar{\varepsilon} y)$$

is defined in $\Omega \times \mathbb{R}^d$, Y–periodic in y. It satisfies (2).

2.2 $\mathcal{D}iscrete$ Extension

Let us assume that we have some finite partition $\overline{\Omega} = \cup_j \overline{\Omega}_j$, $j = 1, \ldots, N$. $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$. For each Ω_j there is a corresponding $\bar{\varepsilon}$–cube $W_j = W(\hat{x}^j)$, $\Omega_j \subseteq W_j$, $\hat{x}^j$ is a center of W_j.

• First for any fixed $x \in \Omega \cap \Omega_j$ we define an auxiliary function $\tilde{a}(x, \cdot)$ at $y \in W_j$:

$$\tilde{a}(x, y) := a_M(y), \qquad y \in W_j.$$

• Secondly we extend it to the whole $\mathbb{R}^d$ periodically – $\tilde{a}(x, y)$ is $\bar{\varepsilon}$–periodic in y.

• Thirdly

$$a(x, y) := \tilde{a}(x, \bar{\varepsilon} y)$$

is defined in $\Omega \times \mathbb{R}^d$, Y–periodic in y. It satisfies (2).

Remark 2.1. *Both extensions are also well defined in $\overline{\Omega} \times Y$ (this will help to show continuity of some properties in $\overline{\Omega}$). For Ω_j much smaller than W_j, $\hat{x}^j \in \Omega_j$ the $\mathcal{D}iscrete$ Extension can be seen as a discretization of the $\mathcal{C}ontinuous$ Extension.*

In order to use the results of convergence and error estimations, one usually needs smoothness of $a(x, y)$. However it is easy to see that $a(x, y)$ are continuous neither in x nor in y (and are properly defined only a.e.). Anyway, in the next sections our goal will be to show that these $a(x, y)$ can be considered as admissible test functions in the sense of two–scale convergence and at least for the second order elliptic equation with highly oscillated (conductivity, permeability) coefficient the standard procedure [1] still works and solutions of $\{\mathcal{P}^{\varepsilon}\}$ converge to the solution of $\mathcal{P}^0$.

Please note that the convergence of the solutions of $\{\mathcal{P}^{\varepsilon}\}$ is important, but it cannot guarantee that the solution of $\mathcal{P}$ can be well–approximated with the help of the solution of the problem $\mathcal{P}^0$. The approximation may fail since $\mathcal{P}^{\bar{\varepsilon}}$ plays a central role in the construction of the sequence and even if the sequence "converges", $\mathcal{P}^0$ may be 'close' to practically useless problems $\mathcal{P}^{\varepsilon}$, for $\varepsilon \ll \bar{\varepsilon}$ but still 'far' from $\mathcal{P}^{\bar{\varepsilon}}$.

2.3 An example of $\mathcal{P}^{\varepsilon}$ for the elliptic problem $\mathcal{P}$

In this example we consider the second order elliptic problem with homogeneous Dirichlet boundary condition as the initial problem

$$\mathcal{P}: \qquad -\nabla \cdot (a_M(x) \nabla u) = f \qquad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0; \tag{3}$$

and the sequence of problems $\{\mathcal{P}^{\varepsilon}\}$ is

$$\mathcal{P}^{\varepsilon}: \qquad -\nabla \cdot (a(x, x/\varepsilon) \nabla u_{\varepsilon}) = f \qquad \text{in } \Omega, \qquad u_{\varepsilon}|_{\partial\Omega} = 0, \tag{4}$$

where $a_M(x) = \{a_M^{ij}(x)\}$ and $a(x, y) = \{a^{ij}(x, y)\}$ are $d \times d$ matrix functions in general case and $a^{ij}(x, y)$ is a two–scale extension of $a_M^{ij}(x)$.

Naturally $a_M(\cdot)$ is required to be bounded and positive definite. Can we expect similar properties for $a(x, x/\varepsilon)$ which are important for verification that $\{\mathcal{P}^{\varepsilon}\}$ is a sequence of solvable problems?

2.4 Properties of $\mathcal{C}ontinuous$ and $\mathcal{D}iscrete$ Extensions inherited from $a_M(x)$

Proposition 2.1. *A property of $a_M(x)$ which is valid for all $x \in \tilde{\Omega}$ is also valid for $a(x, y)$ in $\Omega \times Y$.*

Proof. For both $\mathcal{C}ontinuous$ Extension and $\mathcal{D}iscrete$ Extension there is a mapping $z : \Omega \times Y \longrightarrow \tilde{\Omega}$ that $a(x, y) = a_M(z(x, y))$. □

Corollary 2.1. *Let $\mathcal{M}$ be the mapping $\mathcal{M} : a_M(\cdot) \longrightarrow a(\cdot,\cdot)$. Then*

- *$\mathcal{M}$ is linear.*
- $|a_M(\cdot)| \xrightarrow{\mathcal{M}} |a(\cdot,\cdot)|$.
- $a_M(\cdot)^p \xrightarrow{\mathcal{M}} a(\cdot,\cdot)^p$.
- *if $a_M(x)$ is uniformly bounded, positive definite matrix function in $\tilde{\Omega}$, $a^{ij}_M(\cdot) \xrightarrow{\mathcal{M}} a^{ij}(\cdot,\cdot)$ then $a(x,y)$ is uniformly bounded, positive definite matrix function in $\Omega \times Y$.*

Proof. For example, if $b_M(\cdot) = |a_M(\cdot)|$ then
$b(x,y) = b_M(z(x,y)) = |a_M(z(x,y))| = |a(x,y)|$. Similar with others. □

We note that $\mathcal{M}$ for the $\mathcal{D}iscrete$ Extension has some similarity with the unfolding operator $\mathcal{T}$ [4].

3 Two–scale convergence and admissible test functions

The concept of two–scale convergence was introduced in [11] and further developed in [1]. A recent review of a two–scale convergence in $L^p(\Omega)$ space can be found in [10]. In this section we formulate some results related to two–scale convergence in $L^2(\Omega)$ mainly following [1], but with some modifications of the concept of admissible test function. We will need these results in Section 6.

Definition 3.1. *Let $\mathcal{B}_{TF} = \mathcal{D}(\Omega \times Y)$ be a base space of test functions.*

A function $f(x,y)$ initially defined a.e. in $\Omega \times \overline{Y}$ we can extend to a Y–periodic function in $\Omega \times \mathbb{R}^d$ by periodical repetition, except perhaps the points periodic to ∂Y.

Lemma 3.1. *For any Y–periodic function $\psi(x,y) \in C(\overline{\Omega} \times \overline{Y})$*

$$\lim_{\varepsilon \to 0} \int_\Omega \psi(x, \frac{x}{\varepsilon})\, dx = \int_\Omega \int_Y \psi(x,y)\, dx\, dy \tag{5}$$

Proof. For example see [12]. □

In the following we will deal with sequences $\{u_\varepsilon\}$. u_ε is a pair $(u, \varepsilon) \in L^2(\Omega) \times \mathbb{R}_+$. The sequence $\{u_\varepsilon\}$ is a sequence of pairs $\{(u_n, \varepsilon_n)\}_{n=0}^{\infty}$ where $\{\varepsilon_n\}$ is a fixed sequence of strictly positive numbers tending to zero. "$\lim\limits_{\varepsilon \to 0}$" is the same as "$\lim\limits_{\substack{n\to\infty \\ \varepsilon=\varepsilon_n}}$".

Definition 3.2. *A sequence $\{u_\varepsilon(x)\}$ from $L^2(\Omega)$ is said to be two-scale convergent to a limit $u_0(x,y) \in L^2(\Omega \times Y)$ if*

(i) for all $\psi \in \mathcal{B}_{TF}$:

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \psi(x, \frac{x}{\varepsilon})\, dx = \int_\Omega \int_Y u_0(x,y) \psi(x,y)\, dx\, dy \tag{6}$$

(ii) u_ε is bounded in $L^2(\Omega)$.

We prefer to insure that all two-scale convergent sequences are bounded. Having chosen $\mathcal{B}_{TF}$ somewhat larger, for instance $L^2[\Omega; C_{per}(Y)]$ we would have (i) ⇒ (ii) due to weak convergence of u_ε. We refer to [10] for the discussion of this topic and for the definitions of the functional spaces like $L^2[\Omega; C_{per}(Y)]$.

Remark 3.1. *The Def. 3.2 has sense since the limit $u_0(x, y)$ is unique as an element of $L^2(\Omega\times Y)$ due to density of $\mathcal{B}_{TF}$ in $L^2(\Omega \times Y)$ and at least the following sequences are two-scale convergent:*

1. *If $\phi(x, y) \in C(\overline{\Omega} \times \overline{Y})$ then $u_\varepsilon(x) = \phi(x, x/\varepsilon)$ two-scale converges to $\phi(x, y)$.*
2. *If $u_\varepsilon(x) \to u(x)$ in $L^2(\Omega)$ then $u_\varepsilon(x)$ two-scale converges to $u_0(x, y) = u(x)$.*

Proof. In both cases u_ε is bounded. The first statement is a consequence of Lem. 3.1. To use Lem. 3.1 in the second statement we should approximate $u(x)$ by a smooth function in $L^2(\Omega)$. □

Remark 3.2. *Usually in the definition of the two–scale convergence one uses $\mathcal{B}_{TF} = \mathcal{D}[\Omega; C^\infty_{per}(Y)]$.*

If we want to check that some sequence $\{u_\varepsilon\}$ is two–scale convergent then it is better to have possibly smaller set of test functions ($\mathcal{B}_{TF}$). But if we already know that $\{u_\varepsilon\}$ is two–scale convergent (for example from compactness result, see Cor. 3.1) then it is desirable to be much more free in choosing ψ for (6).

Definition 3.3. *A Y–periodic function $\phi(x, y)$ square integrable in $\Omega\times Y$ with well defined $\phi(x, x/\varepsilon)$ in $L^2(\Omega)$ for all $\varepsilon \in \{\varepsilon_n\}$ is called an admissible test function (ATF) if for all two-scale convergent sequences $\{u_\varepsilon\}$ with a limit $u_0(x, y)$ holds:*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\phi(x, \frac{x}{\varepsilon})\, dx = \int_\Omega \int_Y u_0(x, y)\phi(x, y)\, dx\, dy.$$

Please note that we do not consider ϕ to be an element of $L^2(\Omega \times Y)$ since different representatives $\bar{\phi}(x, y)$, $\bar{\bar{\phi}}(x, y)$ of the same element $\phi \in L^2(\Omega \times Y)$ may have $\bar{\phi}(x, \frac{x}{\varepsilon}) \neq \bar{\bar{\phi}}(x, \frac{x}{\varepsilon})$ in $L^2(\Omega)$, or even $\bar{\phi}(x, \frac{x}{\varepsilon}) \notin L^2(\Omega)$.

Theorem 3.1. *(see [1], Th.1.2.) From any bounded sequence $\{u_\varepsilon\}$ in $L^2(\Omega)$ it is possible to extract a subsequence $\{u'_\varepsilon\}$ and there exists $u_0(x, y) \in L^2(\Omega \times Y)$ so that for all $\phi(x, y) \in L^2[\Omega; C_{per}(Y)]$:*

$$\lim_{\varepsilon\to 0} \int_\Omega u'_\varepsilon(x)\phi(x, \frac{x}{\varepsilon})\, dx = \int_\Omega \int_Y u_0(x, y)\phi(x, y)\, dx\, dy.$$

Corollary 3.1. *From any bounded sequence $\{u_\varepsilon\}$ in $L^2(\Omega)$ it is possible to extract a two-scale convergent subsequence.*

Proof. $\mathcal{B}_{TF} = D(\Omega \times Y) \subset L^2[\Omega; C_{per}(Y)]$. □

Corollary 3.2. *All functions from $L^2[\Omega; C_{per}(Y)]$ are ATF.*

Proof. Let us assume the opposite: $\phi(x, y) \in L^2[\Omega; C_{per}(Y)]$ and $u_\varepsilon(x)$ two-scale converges to $u_0(x, y)$, but there exists $\delta > 0$, subsequence $\{u'_\varepsilon\}$ that

$$|\int_\Omega u'_\varepsilon(x)\phi(x, x/\varepsilon)\, dx - \int_\Omega \int_Y u_0(x, y)\phi(x, y)\, dx\, dy| \geq \delta. \tag{7}$$

From Def. 3.2(ii), Th. 3.1 there exists a subsequence u''_ε in u'_ε that for all $\psi(x, y) \in L^2[\Omega; C_{per}(Y)]$

$$\lim_{\varepsilon\to 0} \int_\Omega u''_\varepsilon(x)\psi(x, \frac{x}{\varepsilon})\, dx = \int_\Omega \int_Y u_1(x, y)\psi(x, y)\, dx\, dy.$$

$u_1(x, y) = u_0(x, y)$ due to the uniqueness of the two-scale limit of $\{u''_\varepsilon\}$. For $\psi = \phi$ there is a contradiction with (7). □

3.1 Necessary and sufficient conditions for ϕ to be ATF

Let us assume that $\phi(x,y) \in \mathcal{A}_{TF}$ – a set of ATF.

- First we test Def. 3.3 with $u_\varepsilon(x) = \psi(x, x/\varepsilon)$, for all $\psi \in \mathcal{B}_{TF}$. $\{u_\varepsilon\}$ two scale converges to $\psi(x,y)$ (see Rem. 3.1)

$$\lim_{\varepsilon \to 0} \int_\Omega \psi(x, \frac{x}{\varepsilon}) \phi(x, \frac{x}{\varepsilon})\, dx = \int_\Omega \int_Y \psi(x,y)\phi(x,y)\, dx\, dy.$$

- Second we test Def. 3.3 with $u_\varepsilon(x) = u(x)$, for all $u(x) \in L^2(\Omega)$ (see Rem. 3.1).

$$\lim_{\varepsilon \to 0} \int_\Omega u(x) \phi(x, \frac{x}{\varepsilon})\, dx = \int_\Omega \int_Y u(x)\phi(x,y)\, dx\, dy.$$

  then $\phi(x, x/\varepsilon)$ weakly converges to $\int_Y \phi(x,y)\, dy$ in $L^2(\Omega)$ and consequently is bounded. From 1 and 2 we conclude that $u_\varepsilon(x) = \phi(x, x/\varepsilon)$ two-scale converges to $\phi(x,y)$.

- Third we test Def. 3.3 with $u_\varepsilon(x) = \phi(x, x/\varepsilon)$:

$$\lim_{\varepsilon \to 0} \int_\Omega \phi(x, \frac{x}{\varepsilon})^2\, dx = \int_\Omega \int_Y \phi(x,y)^2\, dx\, dy.$$

Proposition 3.1. *The necessary conditions for a function ϕ to be from $\mathcal{A}_{TF}$:*

$$\phi(x, x/\varepsilon) \textit{ two-scale converges to } \phi(x,y) \tag{8a}$$

$$\lim_{\varepsilon \to 0} \|\phi(x, x/\varepsilon)\|_{L^2(\Omega)} = \|\phi(x,y)\|_{L^2(\Omega \times Y)} \tag{8b}$$

The conditions implicitly require that $\phi(x,y)$ is square integrable in $\Omega \times Y$ and $\phi(x, x/\varepsilon) \in L^2(\Omega)$ is well-defined for all $\varepsilon \in \{\varepsilon_n\}$.

Theorem 3.2. *Let $u_\varepsilon(x), v_\varepsilon(x) \in L^2(\Omega)$ two-scale converge to $u_0(x,y), v_0(x,y) \in L^2(\Omega \times Y)$ respectively. And also $\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega \times Y)}$ then*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) v_\varepsilon(x)\, dx = \int_\Omega \int_Y u_0(x,y) v_0(x,y)\, dx\, dy.$$

Proof. See the proof of Th.1.8 in [1]. There one can choose $\psi_n(x,y)$ from $\mathcal{B}_{TF} = \mathcal{D}(\Omega \times Y)$, $\phi(x) = 1$ even if 1 is not in $\mathcal{D}(\Omega)$. Note that v_ε must be bounded in assumptions of Th.1.8 and here it is due to (ii) in Def. 3.2. □

Corollary 3.3. *Necessary conditions (8) are also sufficient for a function to be ATF.*

Proof. In Th. 3.2, $u_\varepsilon(x) = \phi(x, x/\varepsilon)$. ϕ satisfies conditions (8). v_ε is an arbitrary two-scale convergent sequence. By Def. 3.3 ϕ is ATF. □

With the help of (8) we can verify whether a particular function is ATF. The condition (8b) alone is not enough [12, Rem. 1.4.5]. Although having a linear space of functions satisfying (8b), there is no need to check (8a):

Proposition 3.2. *Let L be a linear space of functions such that $L \supset \mathcal{B}_{TF}$ and all functions from L satisfy (8b). Then $L \subset \mathcal{A}_{TF}$.*

Proof. We have to check (8a) for $\phi \in L$. $u_\varepsilon(x) = \phi(x, x/\varepsilon)$ is bounded due to (8b). For any $\psi(x, y) \in \mathcal{B}_{TF}$:

$$\phi(x, x/\varepsilon)\psi(x, x/\varepsilon) = \frac{1}{2}\left\{[\phi(x, x/\varepsilon) + \psi(x, x/\varepsilon)]^2 - \phi(x, x/\varepsilon)^2 - \psi(x, x/\varepsilon)^2\right\}$$

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega} \phi(x, \frac{x}{\varepsilon})\psi(x, \frac{x}{\varepsilon})\, dx = \int\limits_{\Omega}\int\limits_{Y} \phi(x, y)\psi(x, y)\, dx\, dy$$

We used (8b) for $\phi + \psi, \phi, \psi \in L$. □

Proposition 3.3. *$\mathcal{A}_{TF}$ is a linear space.*

Proof. Let $\phi_1, \phi_2 \in \mathcal{A}_{TF}$, real numbers α, β. We need to check (8) for $\alpha\phi_1 + \beta\phi_2$. For any $\psi \in \mathcal{B}_{TF}$, (8a) is valid:

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega} \left[(\alpha\phi_1 + \beta\phi_2)\psi\right](x, x/\varepsilon)\, dx = \int\limits_{\Omega}\int\limits_{Y} \left[(\alpha\phi_1 + \beta\phi_2)\psi\right](x, y)\, dx\, dy.$$

We can use (8b) for ϕ_1, ϕ_2 and Th. 3.2 with $u_\varepsilon(x) = \phi_1(x, x/\varepsilon)$, $v_\varepsilon(x) = \phi_2(x, x/\varepsilon)$ to verify (8b) for $\alpha\phi_1 + \beta\phi_2$:

$$[\alpha\phi_1 + \beta\phi_2]^2(x, x/\varepsilon) = [\alpha^2\phi_1^2 + \beta^2\phi_2^2 + 2\alpha\beta\phi_1\phi_2](x, x/\varepsilon).$$

Hence

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega} [\alpha\phi_1 + \beta\phi_2]^2(x, x/\varepsilon)\, dx = \int\limits_{\Omega}\int\limits_{Y} [\alpha\phi_1 + \beta\phi_2]^2(x, y)\, dx\, dy.$$

□

The following sections contain properties of the two–scale $\mathcal{C}ontinuous$ and $\mathcal{D}iscrete$ extensions of a_M respectively. Our main goal is to show that these extensions $a(x, y)$ are ATF. If $a_M \in L^1(\tilde{\Omega})$ then we assume that $a(x, y)$ and $a(x, x/\varepsilon)$ are constructed pointwise a.e. in $\Omega \times Y$ and in Ω from some representative $a_M(x)$ of a_M. Another representative $\bar{a}_M(x)$ results in a.e. the same functions $\bar{a}(x, y)$ and $\bar{a}(x, x/\varepsilon)$.

4 Properties of the two-scale $\mathcal{C}ontinuous$ Extension

In this section we deal only with the extension $a(x, y)$ constructed from $a_M(x)$ in the subsection 2.1.

Proposition 4.1. *For fixed $x \in \Omega$, $a(x, \cdot)$ was constructed piecewise from $a_M(\cdot)$, namely $\mathbb{R}^d$ is divided into 1^d-cubes, by the grid*

$$\mathcal{N}_x(x) = \left\{y \in \mathbb{R}^d \mid \exists k \in \{1, \ldots, d\}, i \in \mathbb{Z} : \quad y_k = x_k/\bar{\varepsilon} + i - 1/2\right\},$$

each cube corresponds to the same $\bar{\varepsilon}$-cube $W(x)$.

Proposition 4.2. *Let us now fix some $y \in \mathbb{R}^d$. The function $a(\cdot, y)$ is piecewise constant on $x \in \Omega$: for each y, Ω is divided by cubic $\bar{\varepsilon}$ grid*

$$\mathcal{N}_y(y) = \{x \in \Omega \mid y \in \mathcal{N}_x(x)\} = \{x \in \Omega \mid \exists k, i : \quad x_k = y_k\bar{\varepsilon} - (i - 1/2)\bar{\varepsilon}\}$$

into parts where $a(\cdot, y)$ is constant.

The way in which $a(x,y)$ was constructed makes it difficult to deal with $a(x,x/\varepsilon)$. We need a simple representation of $a(x,x/\varepsilon)$ for $\varepsilon \neq \bar{\varepsilon}$. The first argument x determines the set $\mathcal{N}_x(x)$ in $\mathbb{R}^d$. The second argument x/ε determines which value of $a_M(x)$ in the neighbourhood $W(x)$ should be taken as the value $a(x,x/\varepsilon)$. The non–periodicity of $a_M(x)$ causes an uncertainty when $x/\varepsilon \in \mathcal{N}_x(x)$.

$$\mathcal{N} = \Big\{x \in \mathbb{R}^d \mid x/\varepsilon \in \mathcal{N}_x(x)\Big\} = \Big\{x \in \mathbb{R}^d \mid \exists k, i : \quad x_k/\varepsilon = x_k/\bar{\varepsilon} + i - 1/2\Big\}$$

$$\mathcal{N} = \Big\{x \in \mathbb{R}^d \mid \exists k \in \{1,\dots,d\}, i \in \mathbb{Z} : x_k = (i - 1/2)\varepsilon\bar{\varepsilon}/\left(\bar{\varepsilon} - \varepsilon\right)\Big\}$$

$\mathcal{N}$ divides $\mathbb{R}^d$ into open cubes $\tilde{\Delta}_I$ with a side $\Delta = \varepsilon\bar{\varepsilon}/|\bar{\varepsilon} - \varepsilon|$ and centers in

$$\dot{x}_I = \frac{\varepsilon\bar{\varepsilon}}{\bar{\varepsilon} - \varepsilon} I, \qquad I = (i_1 \dots i_d) \in \mathbb{Z}^d$$

Let $\mathbf{J}_\varepsilon$ be a set of multiindexes $I \in \mathbb{Z}^d$ that $\Delta_I := \tilde{\Delta}_I \cap \Omega$ is not an empty set.

If $\dot{x}_I \in \Omega$ then

$$a\left(\dot{x}_I, \frac{\dot{x}_I}{\varepsilon}\right) = \tilde{a}\left(\dot{x}_I, \frac{\bar{\varepsilon}}{\varepsilon}\dot{x}_I\right) = \tilde{a}(\dot{x}_I, \dot{x}_I + \bar{\varepsilon} I) = \{\bar{\varepsilon}\text{-periodicity}\} = \tilde{a}(\dot{x}_I, \dot{x}_I) = a_M(\dot{x}_I)$$

Similar if $x \in \Delta_I$ then $x = \dot{x}_I + h$, $|h_k| < \Delta/2$

$$a\left(\dot{x}_I + h, \frac{\dot{x}_I + h}{\varepsilon}\right) = \tilde{a}(\dot{x}_I + h, \frac{\bar{\varepsilon}}{\varepsilon}(\dot{x}_I + h)) = \{\bar{\varepsilon}\text{-periodicity}\} = \tilde{a}(\dot{x}_I + h, \dot{x}_I + \frac{\bar{\varepsilon}}{\varepsilon} h) =$$

$$= a_M(\dot{x}_I + \frac{\bar{\varepsilon}}{\varepsilon} h) \qquad \text{since} \qquad \dot{x}_I + \frac{\bar{\varepsilon}}{\varepsilon} h \in W(\dot{x}_I + h).$$

We have proved the following

Proposition 4.3. *The simple representation of $a(x,x/\varepsilon)$ for all $\varepsilon > 0$, $\varepsilon \neq \bar{\varepsilon}$ is:*

$$\text{if } x \in \Delta_I \qquad \text{then} \qquad a\left(x, \frac{x}{\varepsilon}\right) = a_M(\dot{x}_I + \frac{\bar{\varepsilon}}{\varepsilon}(x - \dot{x}_I))$$

or using the Heaviside function $\mathbf{1}_{\Delta_I}(x)$ being 1 in Δ_I and 0 elsewhere we have:

$$a(x,x/\varepsilon) = \sum_{I \in \mathbf{J}_\varepsilon} \mathbf{1}_{\Delta_I}(x) a_M\big(\dot{x}_I + \frac{\bar{\varepsilon}}{\varepsilon}(x - \dot{x}_I)\big) \tag{9}$$

Roughly speaking for $\varepsilon < \bar{\varepsilon}$ [$\varepsilon > \bar{\varepsilon}$] $a(x,x/\varepsilon)$ is built from compressed [stretched] cubes taken from $a_M(x)$.

For the following let $\phi(x,y)$ be a function from $C(\overline{\Omega} \times \overline{Y})$, Y-periodic in y. We will consider $a(x,y)\phi(x,y)$. Important particular case: $\phi(x,y) = 1$.

Proposition 4.4. *1) If $a_M(x)$ is measurable in $\tilde{\Omega}$ then $a(x,x/\varepsilon)\phi(x,x/\varepsilon)$ is measurable in Ω. 2) If $a_M(x) \in L^1(\tilde{\Omega})$, then $a(x,x/\varepsilon)\phi(x,x/\varepsilon) \in L^1(\Omega)$.*

Proof. We only have to consider the case $\varepsilon \neq \bar{\varepsilon}$.
1) $\phi(x,x/\varepsilon) \in C(\overline{\Omega})$ is measurable. $a(x,x/\varepsilon)$ is measurable since it is a sum of measurable functions (9).

2) If Ω is bounded with the diameter $2R$, then $\bigcup_{I\in\mathbf{J}_\varepsilon}\tilde{\Delta}_I$ is bounded with the diameter $D := 2(R+\sqrt{d}\Delta)$,

$$\sum_{I\in\mathbf{J}_\varepsilon}\Delta^d=\sum_{I\in\mathbf{J}_\varepsilon}\mu(\tilde{\Delta}_I)\le D^d \quad\Rightarrow\quad \sum_{I\in\mathbf{J}_\varepsilon}\frac{\varepsilon^d\bar{\varepsilon}^d}{|\bar{\varepsilon}-\varepsilon|^d}\le D^d \quad\Rightarrow\quad \sum_{I\in\mathbf{J}_\varepsilon}\frac{\varepsilon^d}{\bar{\varepsilon}^d}\le\frac{|\bar{\varepsilon}-\varepsilon|^d}{\bar{\varepsilon}^{2d}}D^d,$$

$$\widetilde{W}(\dot{x}_I):=\left\{z=\dot{x}_I+\frac{\bar{\varepsilon}}{\varepsilon}(x-\dot{x}_I)\mid x\in\Delta_I\right\}=\left\{z=\dot{x}_I+\frac{\bar{\varepsilon}}{\varepsilon}h\mid \dot{x}_I+h\in\Delta_I\right\}\subset\tilde{\Omega}$$

$$\|a(x,x/\varepsilon)\|_{L^1(\Omega)}=\int\limits_{\Omega}|a(x,x/\varepsilon)|\,dx=\sum_{I\in\mathbf{J}_\varepsilon}\int\limits_{\Delta_I}|a_M(\dot{x}_I+\frac{\bar{\varepsilon}}{\varepsilon}(x-\dot{x}_I))|\,dx\le$$

$$\le\sum_{I\in\mathbf{J}_\varepsilon}\frac{\varepsilon^d}{\bar{\varepsilon}^d}\int\limits_{\widetilde{W}(\dot{x}_I)}|a_M(z)|\,dz\le\|a_M\|_{L^1(\tilde{\Omega})}\sum_{I\in\mathbf{J}_\varepsilon}\frac{\varepsilon^d}{\bar{\varepsilon}^d}\le\|a_M\|_{L^1(\tilde{\Omega})}\frac{|\bar{\varepsilon}-\varepsilon|^d}{\bar{\varepsilon}^{2d}}D^d.$$

$$\int\limits_{\Omega}|a(x,x/\varepsilon)\phi(x,x/\varepsilon)|\,dx\le\|\phi\|_C\int\limits_{\Omega}|a(x,x/\varepsilon)|\,dx\le\|\phi\|_C\|a_M\|_{L^1(\tilde{\Omega})}\frac{|\bar{\varepsilon}-\varepsilon|^d}{\bar{\varepsilon}^{2d}}D^d.$$

□

Proposition 4.5. *If $a_M(x)\in L^1(\tilde{\Omega})$ then*

$$M(x)=\int_Y a(x,y)\phi(x,y)\,dy,\quad M_+(x)=\int_Y|a(x,y)\phi(x,y)|\,dy\quad \text{are continuous in } \overline{\Omega},$$

$M(x)$,$M_+(x)$ are bounded by $\|\phi\|_C\|a_M\|_{L^1(\tilde{\Omega})}/\bar{\varepsilon}^d$.

Proof. $a(x,\cdot)\in L^1(Y)$ since it was constructed from $a_M(\cdot)$. Therefore $M(x)$ and $M_+(x)$ are well defined. To show continuity let us fix an arbitrary $E>0$.

$$|M(x+h)-M(x)|=\left|\int_Y a(x+h,y)\phi(x+h,y)\,dy-\int_Y a(x,y)\phi(x,y)\,dy\right|\le$$

$$\left|\int\limits_Y a(x+h,y)[\phi(x+h,y)-\phi(x,y)]\,dy\right|+\left|\int\limits_Y a(x+h,y)\phi(x,y)\,dy-\int\limits_Y a(x,y)\phi(x,y)\,dy\right|$$

For continuous ϕ one can find such δ_1 that $|\phi(x+h,y)-\phi(x,y)|<E\bar{\varepsilon}^d/2\|a_M\|_{L^1(\tilde{\Omega})}$ when $|h|_\infty<\delta_1$ ($|h|_\infty=\max\limits_k|h_k|$ we distinguish from the vector's absolute value $|h|=\sqrt{\sum_k h_k^2}$). This means that the first absolute value is less than $E/2$.
Now we consider the second absolute value. Using that $a(x,y)=\tilde{a}(x,\bar{\varepsilon}y)$, substitution of variables $z=\bar{\varepsilon}y$ we obtain

$$\int_Y a(x,y)\phi(x,y)\,dy=\frac{1}{\bar{\varepsilon}^d}\int_{\bar{\varepsilon}Y}\tilde{a}(x,z)\phi(x,z/\bar{\varepsilon})\,dz=$$

$\tilde{a}(x,z)\phi(x,z/\bar{\varepsilon})$ is $\bar{\varepsilon}$-periodic in z, integral over $\bar{\varepsilon}Y$ is equal to integral over any $\bar{\varepsilon}$ cube

$$=\frac{1}{\bar{\varepsilon}^d}\int_{W(x)}\tilde{a}(x,z)\phi(x,z/\bar{\varepsilon})\,dz=\frac{1}{\bar{\varepsilon}^d}\int_{W(x)}a_M(z)\phi(x,z/\bar{\varepsilon})\,dz$$

Similar

$$\int_Y a(x+h,y)\phi(x,y)\,dy=\frac{1}{\bar{\varepsilon}^d}\int_{W(x+h)}a_M(z)\phi(x,z/\bar{\varepsilon})\,dz$$

$$\frac{1}{\bar{\varepsilon}^d}\Big|\int\limits_{W(x+h)} a_M(z)\phi(x,z/\bar{\varepsilon})\,dz - \int\limits_{W(x)} a_M(z)\phi(x,z/\bar{\varepsilon})\,dz\Big| \le \frac{\|\phi\|_C}{\bar{\varepsilon}^d}\int\limits_{W(x+h)\triangle W(x)} |a_M(z)|\,dz$$

$\mu\left(W(x+h)\triangle W(x)\right) \le 2d\bar{\varepsilon}^{d-1}|h|_\infty$ Using absolute continuity of Lebesgue integral, there exists δ_2: $|h|_\infty < \delta_2$ guarantees that the second absolute value is less than $E/2$ and consequently for $|h|_\infty < \min\{\delta_1,\delta_2\}$ we have $|M(x+h)-M(x)| < E$.

$$2) \qquad |M(x)| \le \frac{\|\phi\|_C}{\bar{\varepsilon}^d}\int_{\tilde{\Omega}} |a_M(x)|\,dx = \|\phi\|_C\|a_M\|_{L^1(\tilde{\Omega})}\Big/\,\bar{\varepsilon}^d.$$

Similar with $M_+(x)$. ☐

Proposition 4.6. *If $a_M(x)$ is measurable in $\tilde{\Omega}$ then $a(x,y)\phi(x,y)$ is measurable in $\Omega\times Y$.*

Proof. $\phi(x,y)$ is continuous hence measurable. To show measurability of $a(x,y)$ we will construct a sequence of measurable functions $\{a_\delta(x,y)\}$ converging to $a(x,y)$ a.e when $\delta\to 0$. Let us divide $\mathbb{R}^d$ into cubes $\square_i^\delta = \Big[i_1\delta,(i_1+1)\delta\Big)\times\cdots\times\Big[i_d\delta,(i_d+1)\delta\Big)$, $i\in\mathbb{Z}^d$. $\mathbf{I}_\delta$ is a set of indexes $i\in\mathbb{Z}^d$ that $\square_i^\delta\cap\Omega\neq\emptyset$. δ is small enough that $\Omega\subset\bigcup\limits_{i\in I_\delta}\square_i^\delta\subset\tilde{\Omega}$. For $i\in\mathbf{I}_\delta$ let $\tilde{x}_i^\delta$ be an arbitrary point of $\square_i^\delta\cap\Omega$ (e.g. the center).

$$\text{The function} \qquad a_\delta(x,y) := a(\tilde{x}_i^\delta, y), \qquad \text{when } x\in\square_i^\delta\cap\Omega,\ i\in I_\delta,$$

is measurable in $\Omega\times Y$ since $a(\tilde{x}_i^\delta,y)$ is a measurable function in $Y=(0,1)^d$ and $\square_i^\delta\cap\Omega$ is a measurable set. We have to show that the sequence pointwise converges to $a(x,y)$ in $\Omega\times Y\setminus O$, where $O := \{(x,y)\in\Omega\times Y \mid y\in\mathcal{N}_x(x)\}$ is a zero measure set.

$$O\cap\Big(\square_i^\delta\times Y\Big)\subset O_i^\delta := \square_i^\delta\times\Big(Y\cap\bigcup_{x\in\square_i^\delta}\mathcal{N}_x(x)\Big), \qquad O\subset\bigcup_{i\in\mathbf{I}_\delta}O_i^\delta.$$

O_i^δ is a measurable set, $\mu_{X\times Y}(O_i^\delta) = \mu_X(\square_i^\delta)\times\mu_Y(Y\cap\bigcup\limits_{x\in\square_i^\delta}\mathcal{N}_x(x)) \le \mu_X(\square_i^\delta)d\frac{\delta}{\bar{\varepsilon}}$.

$$\mu_{X\times Y}(O) \le \sum_{i\in\mathbf{I}_\delta} d\frac{\delta}{\bar{\varepsilon}}\mu_X(\square_i^\delta) \le d\frac{\delta}{\bar{\varepsilon}}\mu_X(\tilde{\Omega}) \to 0, \text{ when } \delta\to 0 \quad\Rightarrow\quad \mu_{X\times Y}(O) = 0.$$

Let $(x,y)\in(\Omega\times Y)\setminus O$. It means that $dist(y,\mathcal{N}_x(x)) > 0$ (here $dist(y,\hat{y}) = |y-\hat{y}|_\infty$). If we consider a δ-partition with $\delta < \bar{\varepsilon}\,dist(y,\mathcal{N}_x(x))$, $x\in\square_i^\delta$ for some i then $(x,y)\notin O_i^\delta$ since for all $\hat{x}\in\square_i^\delta$, y is enough far from $\mathcal{N}_x(\hat{x})$. As we know from Prop. 4.2 $a(\cdot,y)$ is piecewise constant in Ω and it changes value at those $\hat{x}$ that $y\in\mathcal{N}_x(\hat{x})$. The whole set $\square_i^\delta\cap\Omega$ belongs to the cube where $a(\cdot,y)$ is constant. As a result: $\forall\hat{x}\in\square_i^\delta\cap\Omega$, $a(\hat{x},y) = a(\tilde{x}_i^\delta,y)$. On the other hand from the definition of a_δ: $\forall\hat{x}\in\square_i^\delta\cap\Omega$, $a_\delta(\hat{x},y) = a(\tilde{x}_i^\delta,y)$. Consequently for our particular point $(x,y)\in(\square_i^\delta\times Y)\setminus O$ and small enough δ we have $a_\delta(x,y) = a(x,y)$. ☐

Lemma 4.1. *Let $a_M(x)\in L^1(\tilde{\Omega})$; $\phi(x,y)\in C(\overline{\Omega}\times\overline{Y})$, Y-periodic in y; $a(x,y)$ is the Continuous Extension of $a_M(x)$. Then*

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,x/\varepsilon)\phi(x,x/\varepsilon)\,dx = \int_\Omega\int_Y a(x,y)\phi(x,y)\,dx\,dy. \tag{10}$$

Proof. 1. $a(x,y)\phi(x,y)$ is measurable (Prop. 4.6). $M_+(x)$ is continuous (Prop. 4.5). From Fubini's theorem $a(x,y)\phi(x,y) \in L^1(\Omega \times Y)$, the right hand side is well defined.
2. Integrals in the left hand side are well defined (Prop. 4.4).
3. Now we need to check the equality (10). Let $E > 0$ be an arbitrarily small number, for some ε ($\varepsilon \leq \bar{\varepsilon}/2$) we consider a subdivision of Ω (already defined in Prop. 4.3) with 'central' points $\dot{x}_I$:

$$\overline{\Omega} = \bigcup_{I \in \mathbf{J}_\varepsilon} \overline{\Delta_I}; \quad \mathbf{J}_\varepsilon = \mathbf{J}_\varepsilon^{int} \cup (\mathbf{J}_\varepsilon \setminus \mathbf{J}_\varepsilon^{int}); \quad \mathbf{J}_\varepsilon^{int} = \{I \in \mathbf{J}_\varepsilon \mid \tilde{\Delta}_I \subset \Omega\}; \quad \overline{\Omega^{int}} = \bigcup_{I \in \mathbf{J}_\varepsilon^{int}} \overline{\Delta_I}.$$

For not too bad $\partial\Omega$ and small enough ε, $\mu(\Omega \setminus \Omega^{int})$ is arbitrarily small:

$$\mu(\Omega \setminus \Omega^{int}) \leq \sum_{I \in \mathbf{J}_\varepsilon \setminus \mathbf{J}_\varepsilon^{int}} \Delta^d \leq \frac{\bar{\varepsilon}^d E}{5\|\phi\|_C \|a_M\|_{L^1(\tilde{\Omega})}}.$$

We will approximate the integrals over Ω using the integrals over Ω^{int}. Let us estimate the errors in a similar way as it was done in Prop. 4.4, Prop. 4.5:

$$\int_{\Omega \setminus \Omega^{int}} |a(x,x/\varepsilon)\phi(x,x/\varepsilon)|\, dx \leq \|\phi\|_C \sum_{I \in \mathbf{J}_\varepsilon \setminus \mathbf{J}_\varepsilon^{int}} \frac{\varepsilon^d}{\bar{\varepsilon}^d} \int_{\widetilde{W}(\dot{x}_I)} |a_M(z)|\, dz \leq$$

$$\leq \|\phi\|_C \|a_M\|_{L^1(\tilde{\Omega})} \frac{|\bar{\varepsilon} - \varepsilon|^d}{\bar{\varepsilon}^{2d}} \frac{\bar{\varepsilon}^d E}{5\|\phi\|_C \|a_M\|_{L^1(\tilde{\Omega})}} \leq E/5, \qquad \int_{\Omega \setminus \Omega^{int}} |M(x)|\, dx \leq E/5.$$

Since $M(x)$ is continuous in $\overline{\Omega}$, then for small enough ε, the right hand side integral in (10) can be approximated by the sum

$$\int_\Omega \int_Y a(x,y)\phi(x,y)\, dy\, dx \approx \sum_{I \in \mathbf{J}_\varepsilon^{int}} \mu(\Delta_I) \int_Y a(\dot{x}_I, y)\phi(\dot{x}_I, y)\, dy =$$

$$= \sum_{I \in \mathbf{J}_\varepsilon^{int}} \frac{\mu(\Delta_I)}{\bar{\varepsilon}^d} \int_{\bar{\varepsilon}Y} \tilde{a}(\dot{x}_I, z)\phi(\dot{x}_I, \frac{z}{\bar{\varepsilon}})\, dz = \sum_{I \in \mathbf{J}_\varepsilon^{int}} \frac{\mu(\Delta_I)}{\mu(W(\dot{x}_I))} \int_{W(\dot{x}_I)} a_M(z)\phi(\dot{x}_I, \frac{z}{\bar{\varepsilon}})\, dz. \tag{11}$$

with error not greater than $2E/5$.

The integral in the left hand side of (10) can be approximated by

$$\int_\Omega a(x,x/\varepsilon)\phi(x,x/\varepsilon)\, dx \approx \sum_{I \in \mathbf{J}_\varepsilon^{int}} \int_{\Delta_I} a_M\Big(\dot{x}_I + \frac{\bar{\varepsilon}}{\varepsilon}(x - \dot{x}_I)\Big)\phi(x, \frac{x}{\varepsilon}), dx =$$

or in the new variables $z = \dot{x}_I + (x - \dot{x}_I)\bar{\varepsilon}/\varepsilon$, $\quad x = \dot{x}_I + (z - \dot{x}_I)\varepsilon/\bar{\varepsilon}$, $\quad \Delta_I \to \widetilde{W}(\dot{x}_I)$:

$$= \sum_{I \in \mathbf{J}_\varepsilon^{int}} \frac{\varepsilon^d}{\bar{\varepsilon}^d} \int_{\widetilde{W}(\dot{x}_I)} a_M(z)\phi\Big(x(z), \frac{\dot{x}_I}{\varepsilon} + \frac{z - \dot{x}_I}{\bar{\varepsilon}}\Big)\, dz =$$

$$= \sum_{I \in \mathbf{J}_\varepsilon^{int}} \frac{\mu(\Delta_I)}{\mu(\widetilde{W}(\dot{x}_I))} \int_{\widetilde{W}(\dot{x}_I)} a_M(z)\phi\Big(x(z), \frac{z}{\bar{\varepsilon}}\Big)\, dz. \tag{12}$$

The last equality is since $\dot{x}_I(1/\varepsilon - 1/\bar{\varepsilon}) = I$, $\phi(x,y)$ is Y-periodic and $\varepsilon/\bar{\varepsilon} = \mu(\Delta_I)/\mu(\widetilde{W}(\dot{x}_I))$. The approximation error is not greater than $E/5$. Further approximation of (12):

$$\approx \sum_{I \in \mathbf{J}_\varepsilon^{int}} \frac{\mu(\Delta_I)}{\mu(\widetilde{W}(\dot{x}_I))} \int_{\widetilde{W}(\dot{x}_I)} a_M(z)\phi\Big(\dot{x}_I, \frac{z}{\bar{\varepsilon}}\Big)\, dz \tag{13}$$

has an error

$$\sum_{I\in \mathbf{J}_\varepsilon^{int}} \frac{\mu(\Delta_I)}{\mu(\widetilde{W}(\dot{x}_I))} \int_{\widetilde{W}(\dot{x}_I)} a_M(z) \left[\phi\Big(x(z),\frac{z}{\bar\varepsilon}\Big) - \phi\Big(\dot{x}_I,\frac{z}{\bar\varepsilon}\Big)\right]\, dz$$

which we can estimate in absolute value like in Prop. 4.4 (restricting to $\varepsilon \le \bar\varepsilon/2$):

$$\delta\|a_M\|_{L^1(\tilde\Omega)} \sum_{I\in \mathbf{J}_\varepsilon^{int}} \frac{\varepsilon^d}{\bar\varepsilon^d} \le \delta\|a_M\|_{L^1(\tilde\Omega)} \frac{|\bar\varepsilon-\varepsilon|^d}{\bar\varepsilon^{2d}}\mu(\Omega) \le \delta\frac{\|a_M\|_{L^1(\tilde\Omega)}}{\bar\varepsilon^d}\mu(\Omega) \le E/5,$$

where $|\phi(x_1,y)-\phi(x_2,y)| < \delta = E\bar\varepsilon^d/5\|a_M\|_{L^1(\tilde\Omega)}\mu(\Omega)$ when $|x_1-x_2|_\infty \le \Delta/2 \le \varepsilon$.

Now we compare (11) and (13).

$$\text{For } I\in\mathbf{J}_\varepsilon^{int}: \qquad \widetilde{W}(\dot{x}_I) = \{x\in\mathbb{R}^d \mid |x-\dot{x}_I|_\infty \le \frac{\bar\varepsilon^2}{2|\bar\varepsilon-\varepsilon|}\}, \qquad W(\dot{x}_I)\subset \widetilde{W}(\dot{x}_I).$$

$\mu(\widetilde{W}\setminus W) = \mu(\widetilde{W}) - \mu(W) = \bar\varepsilon^{2d}/(\bar\varepsilon-\varepsilon)^d - \bar\varepsilon^d \overset{\varepsilon\to 0}{\longrightarrow} 0$. Then for small enough ε

$$\left|\frac{1}{\mu(\widetilde{W})}\int_{\widetilde{W}} a_M(z)\phi\Big(\dot{x}_I,\frac{z}{\bar\varepsilon}\Big)\,dz - \frac{1}{\mu(W)}\int_W a_M(z)\phi\Big(\dot{x}_I,\frac{z}{\bar\varepsilon}\Big)\,dz\right| \le$$

$$\le \frac{\mu(\widetilde{W})-\mu(W)}{\mu(W)\mu(\widetilde{W})}\int_W |a_M(z)\phi\Big(\dot{x}_I,\frac{z}{\bar\varepsilon}\Big)|\,dz + \frac{1}{\mu(\widetilde{W})}\int_{\widetilde{W}\setminus W} |a_M(z)\phi\Big(\dot{x}_I,\frac{z}{\bar\varepsilon}\Big)|\,dz \le$$

$$\le (\mu(\widetilde{W})-\mu(W))\frac{\|\phi\|_C\|a_M\|_{L^1(\tilde\Omega)}}{\bar\varepsilon^{2d}} + \frac{\|\phi\|_C}{\mu(\widetilde{W})}\int_{\widetilde{W}\setminus W} |a_M(z)|\,dz \le E/5\mu(\Omega).$$

For the second term we used the absolute continuity of Lebesgue integral.

For small enough ε:

$$|\int_\Omega a(x,x/\varepsilon)\phi(x,x/\varepsilon)\,dx - \int_\Omega\int_Y a(x,y)\phi(x,y)\,dy\,dx| \le \frac{4E}{5} + \frac{E}{5\mu(\Omega)}\sum_{I\in\mathbf{J}_\varepsilon^{int}}\mu(\Delta_I) \le E.$$

□

5 Properties of the two-scale $\mathcal{D}iscrete$ Extension

In this section we deal only with the extension $a(x,y)$ constructed from $a_M(x)$ in the Subsection 2.2.

Proposition 5.1. *For fixed $x\in\Omega\cap\Omega_j$, $a(x,\cdot)$ was constructed piecewise from $a_M(\cdot)$, namely $\mathbb{R}^d$ is divided into 1^d cubes by the grid*

$$\mathcal{N}_x(x) = \left\{y\in\mathbb{R}^d \mid \exists k\in\{1\ldots d\}, i\in\mathbb{Z}: \quad y_k = \hat{x}_k^j/\bar\varepsilon + i - 1/2\right\},$$

each cube corresponds to the same $\bar\varepsilon$-cube W_j. If $x_1,x_2\in\Omega_j$ then $\mathcal{N}_x(x_1)=\mathcal{N}_x(x_2)$.

Proposition 5.2. *Let us now fix some $y\in\mathbb{R}^d$. The function $a(\cdot,y)$ is a constant in each Ω_j.*

Like in the previous section, we need a simple representation of $a(x,x/\varepsilon)$. For each W_j if is convenient to correspond an $\bar\varepsilon$-periodic function

$$\tilde a_j(y) := a_M(y) \qquad \text{when } y\in W_j.\ \text{Periodically expanded to } \mathbb{R}^d.$$

$a_j(y) := \tilde a_j(\bar\varepsilon y)$ is 1^d-periodic in $\mathbb{R}^d$.

The simple representation of $a(x,y)$ is

$$a(x,y)=\sum_{j=1}^N \mathbf{1}_{\Omega_j}(x)\,a_j(y), \qquad \text{then} \qquad a(x,x/\varepsilon)=\sum_{j=1}^N \mathbf{1}_{\Omega_j}(x)\,a_j(x/\varepsilon). \tag{14}$$

To describe behaviour of $a_j(x/\varepsilon)$ inside Ω_j let us define the following ε-cubes in $\mathbb{R}^d$

$$\text{with centers in } \dot x_I^j := \hat x^j\varepsilon/\bar\varepsilon + I\varepsilon \qquad \tilde\Delta_I^j=\{x\in\mathbb{R}^d \mid |x-\dot x_I^j|_\infty<\varepsilon/2\}$$

If $x\in\tilde\Delta_I^j$ then $x=\dot x_I^j+h$ ($|h|_\infty<\varepsilon/2$),

$$a_j(x/\varepsilon)=\tilde a_j(\hat x^j+\bar\varepsilon I+h\bar\varepsilon/\varepsilon)=\{\bar\varepsilon\text{-periodicity}\}=\tilde a_j(\hat x^j+h\bar\varepsilon/\varepsilon)=a_M(\hat x^j+h\bar\varepsilon/\varepsilon)$$

since $\hat x^j+h\bar\varepsilon/\varepsilon\in W_j$. We also have a partition of Ω_j:

$$\overline{\Omega}_j=\bigcup_{I\in\mathbf{J}_\varepsilon(j)}\overline{\Delta}_I^j, \qquad I \text{ belongs to } \mathbf{J}_\varepsilon(j) \text{ when } \Delta_I^j:=\tilde\Delta_I^j\cap\Omega_j\neq\emptyset.$$

We have proved the following

Proposition 5.3. *The simple representation of $a(x,x/\varepsilon)$ for all $\varepsilon>0$ is :*

$$a(x,x/\varepsilon)=\sum_{j=1}^N\sum_{I\in\mathbf{J}_\varepsilon(j)}\mathbf{1}_{\Delta_I^j}(x)\,a_M(\hat x^j+\frac{\bar\varepsilon}{\varepsilon}(x-\dot x_I^j))$$

We again assume that $\phi(x,y)\in C(\overline{\Omega}\times\overline{Y})$, Y –periodic in y.

Proposition 5.4. *1)If $a_M(x)$ is measurable in $\tilde\Omega$ then $a(x,x/\varepsilon)\phi(x,x/\varepsilon)$ is measurable in Ω. 2) If $a_M(x)\in L^1(\tilde\Omega)$ then $a(x,x/\varepsilon)\phi(x,x/\varepsilon)\in L^1(\Omega)$.*

Proof. 1) $a_j(x/\varepsilon)$ are measurable; Ω_j are measurable sets; $a(x,x/\varepsilon)$ is a sum of measurable functions. $\phi(x,x/\varepsilon)\in C(\overline{\Omega})$ is measurable.

$$2)\ \int_\Omega|a(x,x/\varepsilon)|\le\sum_{j=1}^N\sum_{I\in\mathbf{J}_\varepsilon(j)}\int_{\tilde\Delta_I^j}|a_M(\hat x^j+\frac{\bar\varepsilon}{\varepsilon}(x-\dot x_I^j))|=\sum_{j=1}^N\sum_{I\in\mathbf{J}_\varepsilon(j)}\left(\frac{\varepsilon}{\bar\varepsilon}\right)^d\int_{W_j}|a_M|\le$$

$$\le\frac{\|a_M\|_{L^1(\tilde\Omega)}}{\bar\varepsilon^d}\sum_{j=1}^N\sum_{I\in\mathbf{J}_\varepsilon(j)}\varepsilon^d=\frac{\|a_M\|_{L^1(\tilde\Omega)}}{\bar\varepsilon^d}\sum_{j=1}^N\mu\Big(\bigcup_{I\in\mathbf{J}_\varepsilon(j)}\tilde\Delta_I^j\Big)\le N\|a_M\|_{L^1(\tilde\Omega)}\left(\frac{\bar\varepsilon+2\varepsilon}{\bar\varepsilon}\right)^d$$

the measures were estimated by $(\bar\varepsilon+2\varepsilon)^d$ since $\Omega_j\subset W_j$ and W_j, $\tilde\Delta_I^j$ have sides $\bar\varepsilon$, ε respectively. Therefore

$$\int_\Omega|a(x,x/\varepsilon)\phi(x,x/\varepsilon)|\,dx\le\|\phi\|_C\int_\Omega|a(x,x/\varepsilon)|\,dx\le N\|\phi\|_C\|a_M\|_{L^1(\tilde\Omega)}\left(\frac{\bar\varepsilon+2\varepsilon}{\bar\varepsilon}\right)^d.$$

$\square$

Proposition 5.5. *If $a_M(x) \in L^1(\tilde{\Omega})$ then*

$$M(x) = \int_Y a(x,y)\phi(x,y)\,dy, \quad M_+(x) = \int_Y |a(x,y)\phi(x,y)|\,dy$$

are continuous in each Ω_j and $M(x)$,$M_+(x)$ are bounded by $\|\phi\|_C \|a_M\|_{L^1(\tilde{\Omega})} \big/ \bar{\varepsilon}^d$.

Proof.

$$M(x) = \int_Y a(x,y)\phi(x,y)\,dy = \int_Y a_j(y)\phi(x,y)\,dy = \frac{1}{\bar{\varepsilon}^d}\int_{\bar{\varepsilon}Y} \tilde{a}_j(z)\phi(x,z/\bar{\varepsilon})\,dz =$$

$$= \{\tilde{a}_j(z)\phi(x,z/\bar{\varepsilon}) \text{ is } \bar{\varepsilon}\text{-periodic in } z\} = \frac{1}{\mu(W_j)}\int_{W_j} a_M(z)\phi(x,z/\bar{\varepsilon})\,dz.$$

Let $E > 0$ be an arbitrarily small number. For $x_1, x_2 \in \Omega_j$, $|\phi(x_1,y) - \phi(x_2,y)| \le \delta = E\bar{\varepsilon}^d / \|a_M\|_{L^1(\tilde{\Omega})}$, when $|x_1 - x_2|_\infty$ is small enough:

$$|M(x_1) - M(x_2)| \le \frac{1}{\mu(W_j)}\int_{W_j} |a_M(z)||\phi(x_1,z/\bar{\varepsilon}) - \phi(x_2,z/\bar{\varepsilon})|\,dz \le E.$$

$$|M(x)| \le \frac{\|\phi\|_C}{\bar{\varepsilon}^d}\int_{W_j} |a_M(z)|\,dz \le \frac{\|\phi\|_C}{\bar{\varepsilon}^d}\int_{\tilde{\Omega}} |a_M(z)|\,dz.$$

Similar with $M_+(x)$. □

Proposition 5.6. *If $a_M(x)$ is measurable in $\tilde{\Omega}$ then $a(x,y)\phi(x,y)$ is measurable in $\Omega \times Y$.*

Proof. $a(x,y)$ is a sum of measurable functions (14), $\phi(x,y)$ is measurable. □

Lemma 5.1. *Let $a_M(x) \in L^1(\tilde{\Omega})$; $\phi(x,y) \in C(\overline{\Omega} \times \overline{Y})$, Y-periodic in y; $a(x,y)$ is the $\mathcal{D}$iscrete Extension of $a_M(x)$. Then*

$$\lim_{\varepsilon \to 0} \int_\Omega a(x,x/\varepsilon)\phi(x,x/\varepsilon)\,dx = \int_\Omega \int_Y a(x,y)\phi(x,y)\,dx\,dy. \tag{15}$$

Proof. Left and right integrals are well defined. Let us fix a small $E > 0$.

$$\int_\Omega \int_Y a(x,y)\phi(x,y)\,dx\,dy = \sum_{j=1}^N \frac{1}{\mu(W_j)}\int_{W_j} a_M(z)\Big[\int_{\Omega_j}\phi(x,z/\bar{\varepsilon})\,dx\Big]\,dz. \tag{16}$$

Let $\mathbf{J}_\varepsilon^{int}(j)$ consists of those $I \in \mathbf{J}_\varepsilon(j)$ that $\tilde{\Delta}_I^j \subset \Omega_j$. $\overline{\Omega}_j^{int} = \bigcup_{I \in \mathbf{J}_\varepsilon^{int}(j)} \overline{\Delta}_I^j$. For not too bad boundaries $\partial\Omega_j$ one can find such small ε that

$$\sum_{j=1}^N \mu(\Omega_j \setminus \Omega_j^{int}) = \sum_{j=1}^N \sum_{I \in \mathbf{J}_\varepsilon(j)\setminus\mathbf{J}_\varepsilon^{int}(j)} \varepsilon^d \le \frac{E\bar{\varepsilon}^d}{2\|\phi\|_C \|a_M\|_{L^1(\tilde{\Omega})}}.$$

Therefore the error of the approximation

$$\int_\Omega a(x,x/\varepsilon)\phi(x,x/\varepsilon)\,dx = \sum_{j=1}^N \sum_{I \in \mathbf{J}_\varepsilon(j)} \int_{\Delta_I^j} a_M(\hat{x}^j + \frac{\bar{\varepsilon}}{\varepsilon}(x - \dot{x}_I^j))\phi(x,x/\varepsilon)\,dx \approx$$

$$\approx \sum_{j=1}^{N} \sum_{I\in \mathbf{J}_\varepsilon^{int}(j)} \int_{\Delta_I^j} a_M(\hat{x}^j + \frac{\bar{\varepsilon}}{\varepsilon}(x - \dot{x}_I^j))\phi(x, x/\varepsilon)\, dx \tag{17}$$

can be estimated in absolute value as

$$\sum_{j=1}^{N} \sum_{I\in \mathbf{J}_\varepsilon(j)\setminus \mathbf{J}_\varepsilon^{int}(j)} \|\phi\|_C \int_{\Delta_I^j} |a_M(\hat{x}^j + \frac{\bar{\varepsilon}}{\varepsilon}(x - \dot{x}_I^j))|\, dx \leq$$

$$\leq \sum_{j=1}^{N} \sum_{I\in \mathbf{J}_\varepsilon(j)\setminus \mathbf{J}_\varepsilon^{int}(j)} \frac{\|\phi\|_C}{\bar{\varepsilon}^d}\varepsilon^d \int\limits_{W_j} |a_M(z)|\, dz \leq \frac{\|\phi\|_C \|a_M\|_{L^1(\tilde{\Omega})}}{\bar{\varepsilon}^d} \sum_{j=1}^{N} \sum_{I\in \mathbf{J}_\varepsilon(j)\setminus \mathbf{J}_\varepsilon^{int}(j)} \varepsilon^d \leq \frac{E}{2}.$$

We continue (17) by introducing new variables for each integral over Δ_I^j: $z = \hat{x}^j + \frac{\bar{\varepsilon}}{\varepsilon}(x - \dot{x}_I^j)$, $x = x_I^j(z) = \dot{x}_I^j + \frac{\varepsilon}{\bar{\varepsilon}}(z - \hat{x}^j)$. Additionally we use that $x/\varepsilon = z/\bar{\varepsilon} + I$ and ϕ is Y–periodic. (17) is equal to

$$\sum_{j=1}^{N} \sum_{I\in \mathbf{J}_\varepsilon^{int}(j)} \Big(\frac{\varepsilon}{\bar{\varepsilon}}\Big)^d \int\limits_{W_j} a_M(z)\phi\Big(x_I^j(z), \frac{z}{\bar{\varepsilon}}\Big)\, dz =$$

$$= \sum_{j=1}^{N} \frac{1}{\mu(W_j)} \int\limits_{W_j} a_M(z)\Big[\sum_{I\in \mathbf{J}_\varepsilon^{int}(j)} \varepsilon^d \phi\Big(x_I^j(z), \frac{z}{\bar{\varepsilon}}\Big)\Big]\, dz$$

and it is approximately equal to (16). If $z \in W_j$ then $x_I^j(z)$ is some point in ε-cube Δ_I^j. For small enough ε the integral from the continuous function ϕ over Ω_j can be approximated by a sum with an error not larger than δ for all $z \in W_j$, $j \in 1, \ldots, N$:

$$|\int_{\Omega_j} \phi\Big(x, \frac{z}{\bar{\varepsilon}}\Big)\, dx - \sum_{I\in \mathbf{J}_\varepsilon^{int}(j)} \varepsilon^d \phi\Big(x_I^j(z), \frac{z}{\bar{\varepsilon}}\Big)| \leq \delta = \frac{E\bar{\varepsilon}^d}{2N\|a_M\|_{L^1(\tilde{\Omega})}}$$

Finally, for small enough ε

$$|\int\limits_{\Omega}\int\limits_{Y} a(x,y)\phi(x,y)\, dy\, dx - \int\limits_{\Omega} a(x, \frac{x}{\varepsilon})\phi(x, \frac{x}{\varepsilon})\, dx| \leq \frac{E}{2} + \sum_{j=1}^{N} \frac{\delta}{\mu(W_j)} \int\limits_{W_j} |a_M(z)|\, dz \leq E.$$

□

6 Admissibility of the $\mathcal{C}ontinuous$, $\mathcal{D}iscrete$ Extensions

Starting from here, if it is not explicitly mentioned, then a two-scale extension means either the $\mathcal{C}ontinuous$ or the $\mathcal{D}iscrete$ Extensions as defined in Subsections 2.1, 2.2.

Corollary 6.1. *If $a_M(\cdot) \in L^p(\tilde{\Omega})$, $p \in \mathbb{N}$, $\phi \in C(\overline{\Omega} \times \overline{Y})$ is Y–periodic, $a(x,y)$ is either the $\mathcal{C}ontinuous$ or the $\mathcal{D}iscrete$ Extension of $a_M(x)$ then*

$$\lim_{\varepsilon\to 0} \int_{\Omega} a(x, x/\varepsilon)^p \phi(x, x/\varepsilon)\, dx = \int_{\Omega}\int_{Y} a(x,y)^p \phi(x,y)\, dx\, dy.$$

Proof. $|a_M|^p = |a_M^p|$ then $b_M(\cdot) = a_M(\cdot)^p \in L^1(\tilde{\Omega})$. As we know from Cor. 2.1 the two-scale extension for $b_M(\cdot)$ is $b(x,y) = a(x,y)^p$. Then we can apply Lem. 4.1 or 5.1 for $b_M(\cdot)$. □

Corollary 6.2. *If $a_M(\cdot) \in L^2(\tilde{\Omega})$, $\psi \in C(\overline{\Omega} \times \overline{Y})$ is Y–periodic then $a(x,y)\psi(x,y)$ is an "admissible" test function .*

Proof. We should check (8a), (8b) for $a(x,y)\psi(x,y)$. $\tilde{\Omega}$ is bounded; therefore $a_M(\cdot) \in L^1(\tilde{\Omega})$. For arbitrary $\phi \in \mathcal{B}_{TF}$ we can choose $p = 1$ and $\phi\psi \in C(\overline{\Omega} \times \overline{Y})$ instead of ϕ in Cor. 6.1 to verify (8a). The second condition (8b) is again a consequence of Cor. 6.1 with $p = 2$ and $\phi = \psi^2 \in C(\overline{\Omega} \times \overline{Y})$. □

7 Application to the elliptic equation

We return back to the practical problem from the Subsection 2.3.

In the context of two-scale convergence, the sequence of problems (4) was investigated in [1],§2. Now what is required is to go through the proofs in order to convince ourselves that they still work in our case when $a(x,y)$ is a two–scale extension of $a_M(x)$.

$$\alpha|\xi|^2 \leq \xi^T a_M(x)\xi, \quad |a_M(x)\xi| \leq \beta|\xi| \qquad \text{in } \tilde{\Omega}, \text{ for any } \xi \in \mathbb{R}^d$$

implies that (Cor. 2.1)

$$\alpha|\xi|^2 \leq \xi^T a(x,y)\xi, \quad |a(x,y)\xi| \leq \beta|\xi| \qquad \text{in } \Omega \times Y, \text{ for any } \xi \in \mathbb{R}^d$$

For any $\varepsilon > 0$, $a(x, x/\varepsilon)$ is measurable in Ω provided $a_M(\cdot)$ is measurable in $\tilde{\Omega}$ (Prop. 4.4, Prop. 5.4, $\phi \equiv 1$). Therefore for $f \in L^2(\Omega)$ the problems (4) are uniquely solvable and their solutions are uniformly bounded in $H_0^1(\Omega)$.

Theorem 7.1. *The sequence u_ε of solutions of (4) converges weakly in $H_0^1(\Omega)$ (and strongly in $L^2(\Omega)$) to u_0, a unique solution of the limit problem:*

$$\mathcal{P}^0: \qquad -\nabla \cdot (A(x)\nabla u_0) = f \qquad \text{in } \Omega, \qquad u_0|_{\partial\Omega} = 0. \tag{18}$$

where

$$A_{ij}(x) = \int_Y e_i^T a(x,y)\Big(\nabla_y w_j(x,y) + e_j\Big)\, dy, \tag{19}$$

e_j - basis vectors, $w_j(x,y)$ $(j = 1, \ldots, d)$ are solutions of the cell problems:

$$\begin{cases} -\nabla_y \cdot \Big(a(x,y)\big(\nabla_y w_j(x,y) + e_j\big)\Big) = 0 \qquad \text{in } Y \\ \int\limits_Y w_j(x,y)\, dy = 0, \qquad w_j(x,y) \text{ is } Y\text{-periodic in } y \end{cases} \tag{20}$$

Proof. For bounded $\tilde{\Omega}$ we have $L^\infty(\tilde{\Omega}) \subset L^2(\tilde{\Omega})$. $\phi(x) \in D(\Omega)$, $\phi_1(x,y) \in D[\Omega; C^\infty_{per}(Y)]$ have continuous derivatives in $\overline{\Omega} \times \overline{Y}$ and according to Cor. 6.2

$$[\nabla\phi(x) + \nabla_y\phi_1(x,y)]^T a(x,y), \quad [\nabla_x\phi_1(x,y)]^T a(x,y)$$

are row vectors consisting of admissible test functions. Therefore it is still possible in this case to pass to the two-scale limit in [1], (2.10) to obtain [1], (2.11). The remaining part is given by [1], Proof of T.2.3.

We note that the uniqueness of the solution to the limit problem resulting in the convergence of the whole sequence (not just some subsequence) is important to insure that the solution to the initial problem $\mathcal{P} = \mathcal{P}^{\bar{\varepsilon}}$ belongs to the convergent sequence. □

What can we say about the averaged coefficient A? In the case of the *Continuous* Extension $a(x,y)$ this coefficient should be calculated at each point $x \in \Omega$ and it depends on the initial coefficient $a_M(\cdot)$ in $\bar{\varepsilon}$-cube $W(x)$ around x. What happens with A if we slightly move from x to $x+h$? For small enough h the volume $W(x)$ has a large intersection with $W(x+h)$ and consequently the coefficients $a(x,\cdot)$, $a(x+h,\cdot)$, which play a crucial role in the cell problem, differ from each other only in a small volume. Therefore continuity of the averaged coefficient depends on the form of cell problem.

Proposition 7.1. *The coefficient $A(x)$ calculated from the Continuous Extension $a(x,y)$ is continuous in $\overline{\Omega}$.*

Proof. Although our cell problem (needed for calculation of $A(x)$) is formulated in Y and has a variational form: find $w_j(x,\cdot) \in H^1_{per}(Y) \setminus \mathbb{R}$ such that

$$\int_Y \nabla_y \phi(y)^T a(x,y) \nabla_y w_j(x,y)\, dy = -\int_Y \nabla_y \phi(y)^T a(x,y) e_j \, dy \qquad \forall \phi \in H^1_{per}(Y) \setminus \mathbb{R},$$

we prefer to deal with the cell problem in terms of $a_M(\cdot)$, not in $a(\cdot,\cdot)$. To do this it is better to substitute $\bar{\varepsilon}$-periodic functions for Y-periodic:

$$\tilde{\phi}(z) := \phi(\frac{z}{\bar{\varepsilon}}), \quad \tilde{w}_j(x,z) := w_j(x, \frac{z}{\bar{\varepsilon}}), \quad \tilde{a}(x,z) = a(x, \frac{z}{\bar{\varepsilon}})$$

After this substitution the integrals will be over $\bar{\varepsilon}Y$ from $\bar{\varepsilon}$-periodic functions in z. Therefore they are equal to integrals over $W(x)$, where $\tilde{a}(x,z) = a_M(z)$.

In new terms the problem has the form: find $\tilde{w}_j(x,\cdot) \in H^1_{per}(W(x)) \setminus \mathbb{R}$ such that for all $\tilde{\phi}(z) \in H^1_{per}(W(x)) \setminus \mathbb{R}$ holds the equality $\mathcal{B}(\tilde{w}_j(x,\cdot), \tilde{\phi}) = \mathcal{L}_j(\tilde{\phi})$, where

$$\mathcal{B}(\tilde{w}, \tilde{\phi}) := \int_{W(x)} \nabla_z \tilde{\phi}(z)^T a_M(z) \nabla_z \tilde{w}(z)\, dz, \quad \mathcal{L}_j(\tilde{\phi}) := -\frac{1}{\bar{\varepsilon}} \int_{W(x)} \nabla_z \tilde{\phi}(z)^T a_M(z) e_j \, dz. \tag{21}$$

The $\bar{\varepsilon}$-periodic function from $H^1_{per}(W(x)) \setminus \mathbb{R}$ is also a function from $H^1_{per}(W) \setminus \mathbb{R}$, where W is an arbitrary $\bar{\varepsilon}$-cube, $\|\phi\|_{H^1_{per}(W(x))\setminus\mathbb{R}} = \|\phi\|_{H^1_{per}(W)\setminus\mathbb{R}}$. The Poincare inequality: for all $\phi \in H^1_{per}(Y) \setminus \mathbb{R}$ and $\tilde{\phi} \in H^1_{per}(W) \setminus \mathbb{R}$

$$\int_Y \phi(y)^2\, dy \le C_\#^2 \int_Y |\nabla_y \phi(y)|^2\, dy, \qquad \int_W \tilde{\phi}(z)^2\, dz \le C_\#^2 \bar{\varepsilon}^2 \int_W |\nabla_z \tilde{\phi}(z)|^2\, dz.$$

We note that the 'small' size of $\bar{\varepsilon}$ is not important here. It is just a fixed constant. The bilinear form $\mathcal{B}$ is elliptic and bounded on $H^1_{per}(W) \setminus \mathbb{R}$:

$$\mathcal{B}(\tilde{w}, \tilde{w}) \ge \alpha \int_{W(x)} |\nabla_z \tilde{w}(z)|^2\, dz \ge \frac{\alpha}{1 + C_\#^2 \bar{\varepsilon}^2} \|\tilde{w}\|^2_{H^1_{per}(W)\setminus\mathbb{R}},$$

$$|\mathcal{B}(\tilde{w}, \tilde{\phi})| \le \beta \|\tilde{w}\|_{H^1_{per}(W)\setminus\mathbb{R}} \|\tilde{\phi}\|_{H^1_{per}(W)\setminus\mathbb{R}}.$$

The linear functional $\mathcal{L}_j$ is bounded on $H^1_{per}(W) \setminus \mathbb{R}$:

$$|\mathcal{L}_j(\tilde{\phi})| \le \beta \bar{\varepsilon}^{d/2-1} \|\tilde{\phi}\|_{H^1_{per}(W)\setminus\mathbb{R}}.$$

Therefore the cell problem has a unique solution $\tilde{w}_j(x,\cdot) \in H^1_{per}(W) \setminus \mathbb{R}$, satisfying:

$$\|\tilde{w}_j(x,\cdot)\|_{H^1_{per}(W)\setminus\mathbb{R}} \le \beta \bar{\varepsilon}^{d/2-1} \frac{1 + C_\#^2 \bar{\varepsilon}^2}{\alpha}$$

The formula (19) written in terms of $\tilde{w}_j$ is

$$A_{ij}(x) = \frac{1}{\bar{\varepsilon}^d} \int_{W(x)} e_i^T a_M(z)(\bar{\varepsilon}\nabla_z \tilde{w}_j(x,z) + e_j)\, dz. \tag{22}$$

x is an arbitrary point from $\overline{\Omega}$. To check the continuity of $A(\cdot)$ we fix some point $x \in \overline{\Omega}$ and consider some point $x+h \in \overline{\Omega}$ to compare $A(x)$ and $A(x+h)$. The cell problem for $A(x+h)$ is

$$\int_{W(x+h)} \nabla_z \tilde{\phi}(z)^T a_M(z) \nabla_z \tilde{w}_j(x+h,z)\, dz = -\frac{1}{\bar{\varepsilon}} \int_{W(x+h)} \nabla_z \tilde{\phi}(z)^T a_M(z) e_j\, dz \tag{23}$$

$S_+ := W(x) \setminus W(x+h)$, $S_- := W(x+h) \setminus W(x)$. $W(x) = W(x+h) \cup S_+ \setminus S_-$. $\int_{W(x)} = \int_{W(x+h)} + \int_{S_+} - \int_{S_-}$. The problem (21) can be re-written:

$$\int_{W(x+h)} \nabla_z \tilde{\phi}(z)^T a_M(z) \nabla_z \tilde{w}_j(x,z)\, dz = -\frac{1}{\bar{\varepsilon}} \int_{W(x)} \nabla_z \tilde{\phi}(z)^T a_M(z) e_j\, dz +$$

$$+ \int_{S_-} \nabla_z \tilde{\phi}(z)^T a_M(z) \nabla_z \tilde{w}_j(x,z)\, dz - \int_{S_+} \nabla_z \tilde{\phi}(z)^T a_M(z) \nabla_z \tilde{w}_j(x,z)\, dz \tag{24}$$

We substitute (23) from (24) denoting $\theta(z) := \tilde{w}_j(x,z) - \tilde{w}_j(x+h,z)$:

$$\int_{W(x+h)} \nabla_z \tilde{\phi}(z)^T a_M(z) \nabla_z \theta(z)\, dz = \hat{\mathcal{L}}(\phi), \tag{25}$$

$$\hat{\mathcal{L}}(\phi) = \frac{1}{\bar{\varepsilon}} \int_{S_-} \nabla_z \tilde{\phi}(z)^T a_M(z) e_j\, dz - \frac{1}{\bar{\varepsilon}} \int_{S_+} \nabla_z \tilde{\phi}(z)^T a_M(z) e_j\, dz +$$

$$+ \int_{S_-} \nabla_z \tilde{\phi}(z)^T a_M(z) \nabla_z \tilde{w}_j(x,z)\, dz - \int_{S_+} \nabla_z \tilde{\phi}(z)^T a_M(z) \nabla_z \tilde{w}_j(x,z)\, dz$$

The point x was fixed. Consequently the function $\tilde{w}_j(x,\cdot)$ is also a fixed function. θ belongs to $H^1_{per}(W) \setminus \mathbb{R}$. Its norm can be estimated by treating (25) as a variational problem for the unknown θ.

$$\left| \frac{1}{\bar{\varepsilon}} \int_{S_+} \nabla_z \tilde{\phi}(z)^T a_M(z) e_j\, dz \right| \leq \beta \|\tilde{\phi}\|_{H^1_{per}(W)\setminus\mathbb{R}} \frac{\sqrt{\mu(S_+)}}{\bar{\varepsilon}}$$

$$\left| \int_{S_+} \nabla_z \tilde{\phi}(z)^T a_M(z) \nabla_z \tilde{w}_j(x,z)\, dz \right| \leq \beta \int_{S_+} |\nabla_z \tilde{\phi}(z)| |\nabla_z \tilde{w}_j(x,z)|\, dz \leq$$

$$\leq \beta \|\nabla_z \tilde{\phi}(z)\|_{[L^2(S_+)]^d} \sqrt{\int_{S_+} |\nabla_z \tilde{w}_j(x,z)|^2\, dz}$$

$|\nabla_z \tilde{w}_j(x,\cdot)|$ is a fixed function from $L^2(W(x))$. Due to the absolute continuity of the Lebesgue integral for arbitrary $E_h > 0$ one can find such $\delta > 0$ that the integral from $|\nabla_z \tilde{w}_j(x,\cdot)|^2$ over any set in $W(x)$ is less than E_h if the set's measure is less than δ. The measure of S_+, is arbitrarily small provided h is small enough.

$$\left| \int_{S_+} \nabla_z \tilde{\phi}(z)^T a_M(z) \nabla_z \tilde{w}_j(x,z)\, dz \right| \leq \beta \|\tilde{\phi}\|_{H^1_{per}(W)\setminus\mathbb{R}} \sqrt{E_h}$$

Similar estimations can be done for the integrals over S_-. Therefore $\|\hat{\mathcal{L}}\|$ and consequently $\|\theta\|_{H^1_{per}(W)\backslash\mathbb{R}}$ are arbitrarily small for small enough h.

$$\|\theta\|_{H^1_{per}(W)\backslash\mathbb{R}} \leq \frac{1 + C_\#^2 \bar{\varepsilon}^2}{\alpha} \|\hat{\mathcal{L}}\|.$$

This helps us to estimate $|A_{ij}(x) - A_{ij}(x+h)|$.

$$A_{ij}(x) - A_{ij}(x+h) = \frac{1}{\bar{\varepsilon}^d} \int_{W(x+h)} e_i^T a_M(z) \bar{\varepsilon} \nabla_z \theta(z)\, dz +$$

$$+\frac{1}{\bar{\varepsilon}^d} \int_{S_+} e_i^T a_M(z)(\bar{\varepsilon}\nabla_z \tilde{w}_j(x,z) + e_j)\, dz - \frac{1}{\bar{\varepsilon}^d} \int_{S_-} e_i^T a_M(z)(\bar{\varepsilon}\nabla_z \tilde{w}_j(x,z) + e_j)\, dz$$

The absolute value of the first term and the integral over S_+ splitted into two parts

$$|\frac{1}{\bar{\varepsilon}^d} \int_{W(x+h)} e_i^T a_M(z) \bar{\varepsilon} \nabla_z \theta(z)\, dz| \leq \beta \|\theta\|_{H^1_{per}(W)\backslash\mathbb{R}} \Big/ \bar{\varepsilon}^{d/2-1},$$

$$|\frac{1}{\bar{\varepsilon}^d} \int_{S_+} e_i^T a_M(z) \bar{\varepsilon} \nabla_z \tilde{w}_j(x,z)\, dz| \leq \beta \sqrt{\mu(S_+) E_h} \Big/ \bar{\varepsilon}^{d-1},$$

$$|\frac{1}{\bar{\varepsilon}^d} \int_{S_+} e_i^T a_M(z) e_j\, dz| \leq \beta \mu(S_+) / \bar{\varepsilon}^d$$

can be made arbitrarily small (due to the small terms $\|\theta\|_{H^1_{per}(W)\backslash\mathbb{R}}$, E_h, $\mu(S_+)$) by choosing small enough h. Together with similar estimations for S_- we have the continuity of $A_{ij}(x)$. □

In the case of the $\mathcal{D}iscrete$ Extension $a(x,y)$, the averaged coefficient $A(x)$ is constant in each Ω_k $(k = 1, \ldots, N)$. To determine it one has to solve N cell problems (21),(22) with $x = \hat{x}^k$ (centers of W_k). This case is realizable in comparison to solving the cell problems at each point in Ω. On the other hand the averaged coefficient being continuous can be interpolated between a finite number of points where it is calculated via cell problems. Here one should be careful since for small $\bar{\varepsilon}$ the averaged coefficient $A(x)$ is as oscillatory as the initial coefficient $a_M(\cdot)$. Increasing $\bar{\varepsilon}$ we expect $A(x)$ to become a function with more and more slow variations and in the subdomains of Ω where the coefficient $a_M(\cdot)$ can be classified as 'spatially homogeneous' it might be close to a constant coefficient.

In Section. 2 we had a restriction on $\bar{\varepsilon}$ from above: $\bar{\varepsilon}$ should be small in comparison with the typical size of Ω. Solving the limit problem numerically with some typical discretization step h provides a restriction for $\bar{\varepsilon}$ from below: roughly speaking, $\bar{\varepsilon}$ should not be smaller than h.

Solving numerically the large number of cell problems is a time consuming task, which can be done in parallel since cell problems are independent from each other and the limit problem. The computational resources can be also saved at least in the following cases:
• $a_M(x)$ has slow variations (for example it can be a constant) in some subdomain $\Omega_{sv} \subset \Omega$. Then inside Ω_{sv} there is no need to average.
• $a_M(x)$ is $\bar{\varepsilon}$-periodic in $\Omega_\# \subset \Omega$ and the directions of periodicity coincide with coordinate axes. Then the constant averaged coefficient inside $\Omega_\#$ can be calculated by solving only one cell problem.

Additionally one can also try to combine this with other types of averaging:
• if the micro coefficient can be classified as statistically homogeneous in some subdomain Ω_{sh} with known averaged value A_{sh} or
• if the averaged coefficient A_{ed} in Ω_{ed} is experimentally determined.
In these cases one can use the coefficients A_{sh} inside Ω_{sh} and A_{ed} in Ω_{ed} instead of solving cell-problems there.

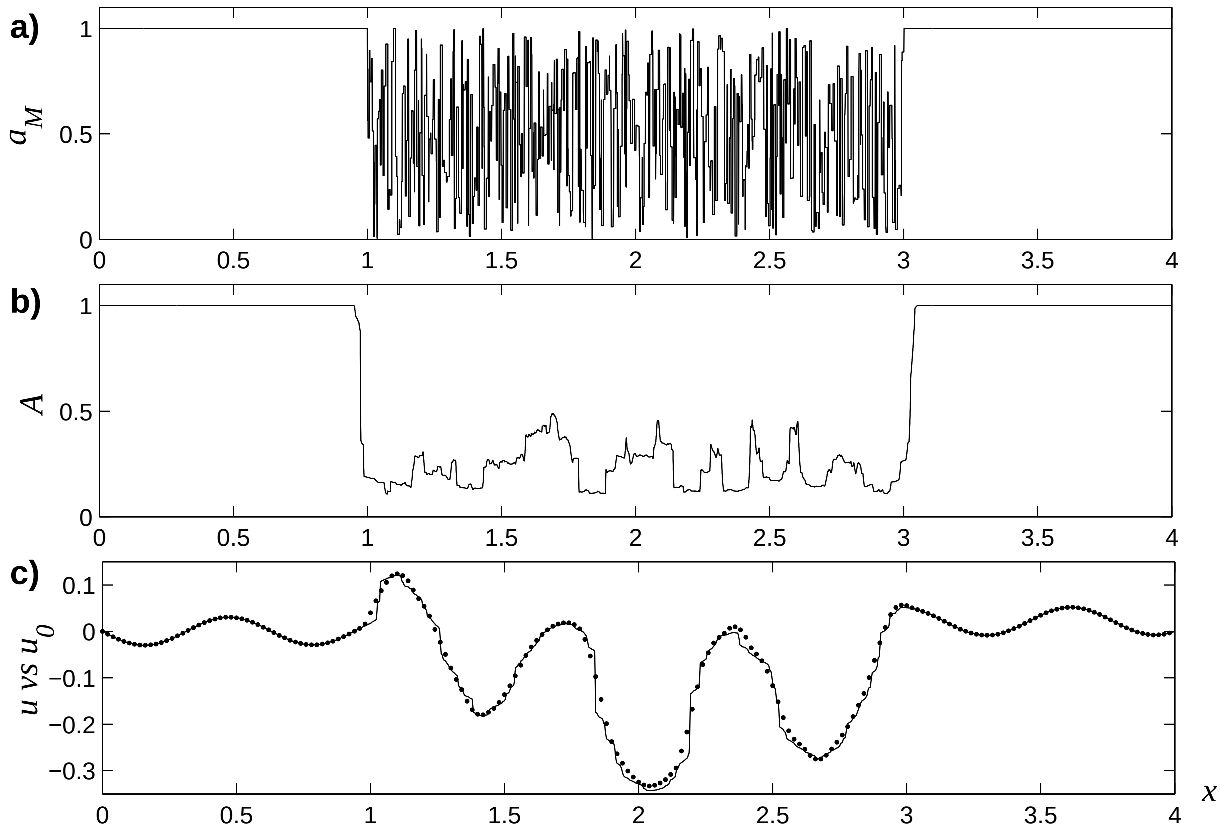

Figure 1: a) initial coefficient $a_M(\cdot)$; b) averaged coefficient $A(\cdot)$; c) comparison $u(x)$ with $u_0(x)$

8 Some concluding remarks

The coefficient $a(x, x/\varepsilon)$ is often used in homogenization as a generalization of the periodic coefficient $a(x/\varepsilon)$. In this paper we propose a way to correspond an averaged (limit) problem for the initial microscopical problem with non-periodic rapidly oscillated coefficient using the results from homogenization together with a special choice of $a(x, y)$ – the two-scale extension of the initial coefficient. The results from homogenization (if not formal) usually require some conditions (like its smoothness) on $a(x, y)$. The lack of smoothness e.g. in the two-scale convergence method can be partially compensated by the "admissibility" of the two-scale extensions, so that e.g. for the second order elliptic equation the convergence of u_ε to u_0 still holds as in the periodic case.

To show that this approach can be useful we present here a 1D example where $a_M(x)$ from (3) and $A(x)$ from (18) are plotted in Fig. 1(a,b) respectively. To calculate $A(\cdot)$, the $\mathcal{C}ontinuous$ extension for $\bar{\varepsilon} = 0.1$ was used. The semi-analytical solutions $u(x)$ (solid line) and $u_0(x)$ (dots) corresponding to $f(x) = -3\sin(10x)$ are compared in Fig. 1(c).

In a 2D test presented in [9] a fine scale reference solution to $\mathcal{P}$ is compared with a H^1-corrected coarse solution to $\mathcal{P}^0$ (this classical correction is described e.g. in [2, p.76]). $A(x)$ in $\mathcal{P}^0$ is calculated via the $\mathcal{D}iscrete$ extension of a randomly generated smooth 2D function $a_M(\cdot)$.

In consequent publications we are planning to present numerical results for the elliptic problem in 1D and 2D more systematically together with some other two-scale extensions.

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